Project systems theory – Solutions

Resit exam 2018–2019, Friday 12 April 2019, 14:00 – 17:00

Problem 1

(6 + 8 + 4 = 18 points)

Consider the model of population dynamics given by

$$\begin{aligned} \dot{x}_1(t) &= \big(\beta_1 - F(x(t))\big)x_1(t), \\ \dot{x}_2(t) &= \big(\beta_2 - F(x(t))\big)x_2(t), \\ \dot{x}_3(t) &= \big(\beta_3 - F(x(t))\big)x_3(t), \end{aligned} \tag{1}$$

where $x_i(t) \in \mathbb{R}$, $i = 1, 2, 3$ denote the populations of three species and $x = [\, x_1 \;\; x_2 \;\; x_3 \,]^{\mathrm{T}}$. Here,

$$F(x) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3, \tag{2}$$

for real parameters $\alpha_i > 0$, $i = 1, 2, 3$, denotes the total burden on the environment and $\beta_i > 0$, $i = 1, 2, 3$, denote the natural growth rates for each species. They are assumed to satisfy

$$\beta_1 > \beta_2 > \beta_3 > 0. \tag{3}$$

(a) Equilibrium points $\bar{x} = [\, \bar{x}_1 \;\; \bar{x}_2 \;\; \bar{x}_3 \,]^{\mathrm{T}}$ are defined as solutions of

$$\begin{aligned} 0 &= \big(\beta_1 - F(\bar{x})\big)\bar{x}_1, \\ 0 &= \big(\beta_2 - F(\bar{x})\big)\bar{x}_2, \\ 0 &= \big(\beta_3 - F(\bar{x})\big)\bar{x}_3. \end{aligned} \tag{4}$$

Starting with the first equation, we consider the cases $\bar{x}_1 \neq 0$ and $\bar{x}_1 = 0$.

Assume that $\bar{x}_1 \neq 0$, then we necessarily have $\beta_1 = F(\bar{x})$. However, as the values of β_i are distinct (see (3)), this means that $\beta_2 - F(\bar{x}) \neq 0$ and $\beta_3 - F(\bar{x}) \neq 0$. Consequently, from the final two equations we obtain $\bar{x}_2 = \bar{x}_3 = 0$. Then, the substitution of this in $\beta_1 = F(\bar{x})$ shows that $\bar{x}_1$ necessarily equals β_1/α_1 (and note that this does not contradict our assumption $\bar{x}_1 \neq 0$). We have thus found the equilibrium point

$$\bar{x} = \begin{bmatrix} \frac{\beta_1}{\alpha_1} \\ 0 \\ 0 \end{bmatrix}. \tag{5}$$

Now, consider the alternative case $\bar{x}_1 = 0$. Then, we either have the trivial equilibrium point $\bar{x} = 0$ or, alternatively, either $\bar{x}_2 \neq 0$ or $\bar{x}_3 \neq 0$ (or both). In the latter case, following the same reasoning as above, we observe that any equilibrium point is of the form $\bar{x}_i > 0$ for some $i = 1, 2, 3$ and $\bar{x}_j$ for $j \neq i$ as desired.

(b) To find the linearized dynamics around the equilibrium (5), introduce the deviation from the equilibrium as

$$\tilde{x} = x - \bar{x}. \tag{6}$$

Then, after introducing the notation

$$f(x) = \begin{bmatrix} (\beta_1 - F(x))x_1 \\ (\beta_2 - F(x))x_2 \\ (\beta_3 - F(x))x_3 \end{bmatrix} = \begin{bmatrix} \beta_1 x_1 - \alpha_1 x_1^2 - \alpha_2 x_1 x_2 - \alpha_3 x_1 x_3 \\ \beta_2 x_2 - \alpha_1 x_1 x_2 - \alpha_2 x_2^2 - \alpha_3 x_2 x_3 \\ \beta_3 x_3 - \alpha_1 x_1 x_3 - \alpha_2 x_2 x_3 - \alpha_3 x_3^3 \end{bmatrix}, \tag{7}$$

we have that

$$\dot{\tilde{x}} = \dot{x} = f\big(\bar{x} + \tilde{x}\big) \approx f(\bar{x}) + \frac{\partial f}{\partial x}(\bar{x})\tilde{x} = \frac{\partial f}{\partial x}(\bar{x})\tilde{x}. \tag{8}$$

A direct computation leads to

$$\frac{\partial f_i}{\partial x_j} = \begin{cases} \beta_i - F(x) - \alpha_i x_i & , \ i = j, \\ -\alpha_j x_i & , \ i \neq j. \end{cases} \tag{9}$$

such that

$$\frac{\partial f}{\partial x}(\bar{x}) = \begin{bmatrix} -\beta_1 & -\frac{\alpha_2}{\alpha_1}\beta_1 & -\frac{\alpha_3}{\alpha_1}\beta_1 \\ 0 & \beta_2 - \beta_1 & 0 \\ 0 & 0 & \beta_3 - \beta_1 \end{bmatrix}. \tag{10}$$

Thus, the linearized dynamics is given as

$$\dot{\tilde{x}}(t) = \begin{bmatrix} -\beta_1 & -\frac{\alpha_2}{\alpha_1}\beta_1 & -\frac{\alpha_3}{\alpha_1}\beta_1 \\ 0 & \beta_2 - \beta_1 & 0 \\ 0 & 0 & \beta_3 - \beta_1 \end{bmatrix} \tilde{x}(t). \tag{11}$$

(c) Stability is determined by the eigenvalues of the matrix (10). Due to the upper block-triangular structure, the eigenvalues are

$$-\beta_1, \quad \beta_2 - \beta_1, \quad \beta_3 - \beta_1. \tag{12}$$

Given the condition (3), these eigenvalues are all (real and) negative, such that the linearized system is (asymptotically) stable.

Problem 2 (4 + 12 + 6 = 22 points)

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{13}$$

with state $x(t) \in \mathbb{R}^3$, input $u(t) \in \mathbb{R}$, and where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{14}$$

(a) Controllability can be verified by evaluating the controllability matrix as

$$\begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & -3 & 8 \end{bmatrix}, \tag{15}$$

whose rank can be seen to equal 3 (note the triangular structure). As this equals the dimension of the state space, the system (13)–(14) is controllable.

(b) As a first step, computation of the characteristic polynomial of A in (14) yields

$$\begin{aligned} \Delta_A(\lambda) = \det(\lambda I - A) &= \begin{vmatrix} \lambda - 1 & -1 & 0 \\ 0 & \lambda + 1 & -1 \\ 0 & 1 & \lambda + 3 \end{vmatrix}, \\ &= (\lambda - 1)\big((\lambda + 1)(\lambda + 3) + 1\big), \\ &= (\lambda - 1)(\lambda^2 + 4\lambda + 4), \\ &= \lambda^3 + 3\lambda^2 - 4. \end{aligned} \tag{16}$$

Hence, after defining

$$a_1 = 3, \quad a_2 = 0, \quad a_3 = -4, \tag{17}$$

we can write

$$\Delta_A(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3, \tag{18}$$

which is of the same form as in the lecture notes.

To find the transformation matrix T, consider

$$q_3 = B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{19}$$

$$q_2 = AB + a_1B = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \tag{20}$$

$$q_1 = A^2B + a_1AB + a_2B = \begin{bmatrix} 1 \\ -4 \\ 8 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} + 0 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \tag{21}$$

and note that the matrix-vector products AB and A^2B are already given in (15). Then, the matrix T defined as

$$T = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \tag{22}$$

has the desired properties. Namely, using the property

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \tag{23}$$

one can verify that

$$T^{-1}AT = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}, \tag{24}$$

and

$$T^{-1}B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{25}$$

as desired. Note that this gives

$$\alpha_1 = -a_3 = 4, \quad \alpha_2 = -a_2 = 0, \quad \alpha_3 = -a_1 = -3, \tag{26}$$

which could have been concluded immediately from (17)–(18) as the standard controllable canonical form (24) is guaranteed by construction.

(c) The matrices $A + BF$ and

$$T^{-1}(A + BF)T = T^{-1}AT + T^{-1}BFT \tag{27}$$

have the same eigenvalues by similarity transformation. Denote

$$\bar{F} = \begin{bmatrix} \bar{f}_3 & \bar{f}_2 & \bar{f}_1 \end{bmatrix} = FT \tag{28}$$

and compute

$$T^{-1}AT + T^{-1}BFT = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \bar{f}_3 - a_3 & \bar{f}_2 - a_2 & \bar{f}_1 - a_1 \end{bmatrix}. \tag{29}$$

Due to its companion form, the characteristic equation of this matrix is easily given as

$$\Delta_{T^{-1}(A+BF)T}(\lambda) = \lambda^3 + (a_1 - \bar{f}_1)\lambda^2 + (a_2 - \bar{f}_2)\lambda + (a_3 - \bar{f}_3). \tag{30}$$

We would like this closed-loop system matrix to have eigenvalues at -2, -2, and -3, such that its desired characteristic polynomial $p(\lambda)$ is given as

$$p(\lambda) = (\lambda + 2)^2(\lambda + 3) = (\lambda^2 + 4\lambda + 4)(\lambda + 3) = \lambda^3 + 7\lambda^2 + 16\lambda + 12. \tag{31}$$

Equating (30) and (31), hereby using the values of a_i, $i \in \{1, 2, 3\}$ in (17), gives

$$\begin{aligned} \bar{f}_1 &= a_1 - 7 = 3 - 7 = -4, \\ \bar{f}_2 &= a_2 - 16 = 0 - 16 = -16, \\ \bar{f}_3 &= a_3 - 12 = -4 - 12 = -16, \end{aligned} \tag{32}$$

such that

$$\bar{F} = \begin{bmatrix} -16 & -16 & -4 \end{bmatrix}. \tag{33}$$

To find the feedback matrix F (in the original coordinates), solve the linear system $FT = \bar{F}$ as

$$F \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -16 & -16 & -4 \end{bmatrix}, \tag{34}$$

which yields

$$F = \begin{bmatrix} -36 & -16 & -4 \end{bmatrix}. \tag{35}$$

Note that the triangular structure of T allows for conveniently solving (34). Alternatively, one could directly compute $F = \bar{F}T^{-1}$ with T^{-1} as in (23).

Problem 3 (15 points)

Consider the linear system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2b & -b & -a \end{bmatrix} x(t), \tag{36}$$

where $a, b \in \mathbb{R}$.

After denoting

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2b & -b & -a \end{bmatrix}, \tag{37}$$

and observing that this matrix is in so-called companion form, it follows that its characteristic polynomial reads

$$\Delta_A(s) = s^3 + as^2 + bs + 2b. \tag{38}$$

We will use the Routh-Hurwitz criterion to assess stability of the polynomial and, hence, of the linear system (36). To this end, consider the following table:

	s^3	s^2	s^1	s^0	
$a\times$	1	a	b	$2b$	(step 0)
$1\times$	a		$2b$		
$(a-2)b\times$		a^2	$(a-2)b$	$2ab$	(step 1)
$a^2\times$		$(a-2)b$			
			$(a-2)^2b^2$	$2ab^2(a-2)$	
			$(a-2)b$	$2ab$	(step 2, after division by $(a-2)b$)

The polynomial indicated in step 0 above is the characteristic polynomial Δ_A. By the Routh-Hurwitz criterion, a necessary condition for stability is that the coefficients corresponding to the two highest powers have the same sign, which means that necessarily $a > 0$. Moreover, a necessary condition for a polynomial to be stable is that all its coefficients have the same sign, which also leads to $b > 0$. Thus, we have

$$a > 0, \qquad b > 0. \tag{39}$$

After the first application of the recursive Routh-Hurwitz theorem, we obtain the polynomial of step 1. Similar to before, a necessary condition for stability is that all coefficients are positive. This strengthens the conditions (39) to

$$a > 2, \qquad b > 0. \tag{40}$$

Then, after the second application of the Routh-Hurwitz theorem, the linear polynomial

$$(a-2)bs + 2ab \tag{41}$$

is obtained. Given the conditions (40), it is readily checked that this polynomial is stable. Hence, the linear system (36) is stable if and only if (40) holds.

Problem 4 (4 + 4 + 4 + 8 = 20 points)

Consider the linear system

$$\dot{x}(t) = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -4 & 7 \\ 1 & -4 & 6 \end{bmatrix} x(t), \qquad y(t) = \begin{bmatrix} 1 & -2 & 2 \end{bmatrix} x(t). \tag{42}$$

(a) To verify whether the system is observable, compute

$$\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \\ 1 & -2 & 2 \end{bmatrix}. \tag{43}$$

As this matrix has rank 1 (note that all rows are linearly dependent), the system is not observable.

(b) The unobservable subspace $\mathcal{N}$ reads

$$\mathcal{N} = \ker \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \ker \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \\ 1 & -2 & 2 \end{bmatrix}, \tag{44}$$

for which a basis is given as

$$\mathcal{N} = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}. \tag{45}$$

In the remainder of this problem, consider the linear system

$$\dot{x}(t) = \begin{bmatrix} a-3 & 8-2a \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 2a \\ a \end{bmatrix} u(t), \tag{46}$$

with a a real parameter.

(c) To determine whether (46) is controllable, denote

$$A = \begin{bmatrix} a-3 & 8-2a \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2a \\ a \end{bmatrix} \tag{47}$$

and compute

$$\begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 2a & 2a \\ a & a \end{bmatrix}, \tag{48}$$

which has rank 1 for $a \neq 0$ and is the zero matrix otherwise. Hence, the system (46) is never controllable.

(d) Recall that the pair (A, B) is stabilizable if and only if

$$\operatorname{rank} \begin{bmatrix} \lambda I - A & B \end{bmatrix} = n \quad \text{for all} \quad \lambda \in \sigma(A) \text{ s.t. } \Re(\lambda) \geq 0. \tag{49}$$

Note that the eigenvalues of A in (47) are given by

$$\lambda_1 = a - 3, \qquad \lambda_2 = 1. \tag{50}$$

First, starting with $\lambda_2 = 1$, we have

$$\begin{bmatrix} \lambda_2 I - A & B \end{bmatrix} = \begin{bmatrix} 4-a & 2a-8 & 2a \\ 0 & 0 & a \end{bmatrix}, \tag{51}$$

which has rank 2 for all a such that $a \neq 0$ and $a \neq 4$ (then, λ_2 is a controllable eigenvalue).

Next, considering $\lambda_1 = a - 3$, it is clear that $a < 3$ implies that $\Re(\lambda_1) < 0$ and there is no need to verify controllability of the eigenvalue. Thus, combining this with the earlier observation on λ_2, we have that (A, B) is stabilizable if

$$a < 3, \qquad a \neq 0. \tag{52}$$

Now, take $a \geq 3$. Then,

$$\begin{bmatrix} \lambda_1 I - A & B \end{bmatrix} = \begin{bmatrix} 0 & 2a - 8 & 2a \\ 0 & a - 4 & a \end{bmatrix} = \begin{bmatrix} 0 & 2(a-4) & 2a \\ 0 & a - 4 & a \end{bmatrix}, \tag{53}$$

which has rank 1. Thus, the eigenvalue λ_1 is never controllable.

Combining these results, we have that (46) is stabilizable if and only if (52) holds.

Problem 5

(15 points)

Consider the linear system

$$\dot{x}(t) = Ax(t), \qquad y(t) = Cx(t), \tag{54}$$

with $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, and A and C real matrices. Recall that a matrix $X \in \mathbb{R}^{n\times n}$ is called positive definite if $v^* X v > 0$ for all $v \neq 0$ and with v^* the Hermitian transpose of v.

Assume that the matrix pair (A, C) is observable and that there exists a positive definite symmetric matrix X that solves the matrix equation

$$A^{\mathrm{T}} X + XA + C^{\mathrm{T}} C = 0. \tag{55}$$

To show that this implies asymptotic stability, let λ be an eigenvalue of A, i.e., $Av = \lambda v$ for some $v \neq 0$. Then, pre- and post-multiplication of (55) by v^* and v, respectively, leads to

$$\begin{aligned} 0 &= v^* A^{\mathrm{T}} X v + v^* X A v + v^* C^{\mathrm{T}} C v, \\ &= \bar{\lambda} v^* X v + \lambda v^* X v + v^* C^{\mathrm{T}} C v, \\ &= 2\Re(\lambda) v^* X v + v^* C^{\mathrm{T}} C v. \end{aligned} \tag{56}$$

Here, $\bar{\lambda}$ denotes the complex conjugate of λ and we have used that $\lambda + \bar{\lambda} = 2\Re(\lambda)$. Then, we obtain

$$\Re(\lambda) = -\frac{v^* C^{\mathrm{T}} C v}{2 v^* X v} \leq 0, \tag{57}$$

where we have used that X is positive definite and that $v^* C^{\mathrm{T}} C v \geq 0$.

This result can be strengthened by noting that $\Re(\lambda) = 0$ if and only if $Cv = 0$. In this case, note that

$$\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} v = 0, \tag{58}$$

which implies

$$\operatorname{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} < n. \tag{59}$$

However, this contradicts (by the Hautus test) the assumption that the matrix pair (A, C) is observable. Thus, we have that $Cv \neq 0$ and, moreover,

$$\Re(\lambda) = -\frac{v^* C^{\mathrm{T}} C v}{2 v^* X v} < 0. \tag{60}$$

This proves the desired result.